On the Evolutionary Stability of Rational Expectations

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Abstract

Evolutionary game theory provides a fresh perspective on the prospects that agents with heterogeneous expectations might eventually come to agree on a single expectation corresponding to the efficient markets hypothesis. We establish conditions where agreement on a unique forecast is stable, but also show that persistent heterogeneous expectations can arise if those conditions do not hold. The critical element is the degree of curvature in payoff weighting functions agents use to value forecasting performance. We illustrate our results in the context of an asset pricing model where a martingale solution competes with the fundamental solution for agents’ attention.

Keywords: rational expectations, heterogeneous expectations, evolutionary game theory, asset pricing, efficient markets hypothesis

JEL codes: C73, D84, G12, C22
1 Introduction

The existence of a representative, rational forecast is a common assumption in the macroeconomic and financial economics literatures. The widespread use of the strong version of the efficient markets (EMH) hypothesis is a prominent example. One challenge to models with forward-looking expectations is explaining how all agents come to hold a single expectation. A lengthy debate has taken place, for example, over how agents seeking to adopt rational expectations might single out the fundamental solution from among the infinite number of martingale solutions.\(^1\) More generally, econometrics texts describe a wide range of forecasting strategies (ARIMA models, for example) that may or may not exemplify rational expectations.\(^2\) Agents in the financial markets pursue an even wider range of “technical” forecasting strategies.\(^3\) This diversity of empirical forecasting procedures is at odds with the common assumption of a representative forecast satisfying rational expectations and the strong version of the efficient markets hypothesis, where asset prices are solely determined by expected future dividends.

Using evolutionary game theory, we attempt to reconcile the theoretical assumption of a single expectation with the empirical reality of multiple forecasting procedures. We establish conditions under which agents agree on a fundamental forecast corresponding to the EMH. This result is not entirely conclusive in justifying representative agent models, however, because we also show that alternative conditions lead to persistent heterogeneous expectations.

The game theoretic basis for our approach has several key features: (i) expectations can be heterogeneous, (ii) agents are allowed to switch strategies based on observed forecasting performance, (iii) convergence to a single expectation is possible, but not necessary, (iv) agents consider a strategy we refer to as reflective that is a weighted average of other forecasts, and (v) one forecast is regarded as fundamental on the basis of economic theory by at least some small fraction of the agents.

The majority of the macroeconomic literature, including the work on rational bubbles (Blanchard (1979), Evans (1991), Charemza and Deadman (1995)), assumes all agents share a unique rational expectation. Within the learning literature, expectations are often homogeneous as well,

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\(^1\)See, for example, Cochrane (2001) and McCallum (1983, 1997).
\(^2\)The econometrics literature in the tradition of Box and Jenkins (1970) specifically focuses on the advantages of empirical models over theory-based models.
\(^3\)Covel (2004), for example, puts forward the merits of “trend following.”
as with the analysis of least squares learning in Marcet and Sargent (1989a) or studies on gradient learning (Evans, Honkapohja and Williams 2005). There are examples of heterogeneous expectations with fixed fractions of agents having idiosyncratic information in both the finance literature, Constanides and Duffie (1996) for example, and in the learning literature (Marcet and Sargent (1989b) and Evans and Guesnerie (2005)) including studies of agents learning sunspot solutions (Branch and McGough (2004)).

Our approach includes heterogeneous forecasts and allows for dynamic switching between forecasting strategies based on past performance. We present an asset pricing model where payoffs to forecasting strategies are based on negative squared forecast errors, following common methods for evaluation of forecasts in the time series literature, see Elliot and Timmerman (2008).

An evolutionary game theory mechanism describes switching between strategies while allowing for the possibility that agents will come to agree on a single forecast. Suppose fractions $x_{1,t}, \ldots, x_{n,t}$ of the agents follow $n$ strategies with forecasts $e_{1,t}, \ldots, e_{n,t}$ that produce payoffs $\pi_{1,t}, \ldots, \pi_{n,t}$. To allow for the possibility that agents agree on a single forecast, we study dynamics of the general form\footnote{The timing of the equation is complicated by since the payoffs in time $t$ depend on the forecast in the previous period.}

$$x_{i,t+1} - x_{i,t} = x_{i,t-1} \frac{w(\pi_{i,t}) - \bar{w}_t}{\bar{w}_t},$$

(1)

where the \textit{weighting function} $w(\cdot)$ is increasing in the payoffs, and the expression $\bar{w}_t$ is the population average weighted according to the popularity of the strategies so that

$$\bar{w}_t = x_{1,t-1}w(\pi_{1,t}) + x_{2,t-1}w(\pi_{2,t}) + \cdots + x_{n,t-1}w(\pi_{n,t}).$$

Strategies with above average payoffs gain adherents. Dynamics such as (1) with $x_{i,t-1}$ on the right-hand side are also \textit{imitative} in that the popularity of a strategy affects its popularity next period. The timing in (1) differs from the customary replicator dynamic. We make this choice since the time $t$ payoffs depend on a forecast made in time $t - 1$, which in turn depend on the time $t - 1$ fractions of followers of the strategies $x_{i,t-1}$. Simulations with an alternative timing are conducted as a robustness check.

There is a substantial literature with dynamic switching of forecasting strategies using the multinomial logit model, particularly with the cobweb model (Brock and Hommes (1997), Hommes (2006)) and with asset pricing\footnote{Branch and Evans (2007) study dynamic switching with multinomial logit in a Lucas-style macro model. Horst and Wenzelburger (2008) examine the long-run behavior of a related asset pricing model.} (Brock and Hommes (1998) and Föllmer, Horst, and Kirman (2005)).
Hommes and Anufriev (2009) estimate a multinomial logit model with multiple forecasting strategies using experimental asset market data, arguing that the data can only be explained through the consideration of heterogeneous forecasts. We choose the imitative dynamic (1) since the outcome where $x_{j,t} = 0$ is a distinct possibility but is not a natural outcome under multinomial logit. Sandholm (2007) and Waters (2009) provide a detailed discussion comparing dynamics.

Blume and Easley (1992) is a prominent example from a related literature that studies the long run survival of investment strategies. These models do not specify the fractions of followers of different strategies each period but focus on the limiting ratios of payoffs between strategies. Hens and Schenk-Hoppe (2009) review recent developments.

If $w(\cdot)$ is linear, equation (1) corresponds to the replicator dynamic, which has been analyzed in detail in the game theory literature, see Weibull (1997). Sethi and Franke (1995) and Branch and McGough (2008) apply alternative versions of the replicator dynamic to the cobweb model and show conditions where chaos arises. All the referenced models studying the cobweb model have agents using a naive forecast based on a simple backward looking forecast and a rational forecast for which they must pay a cost. In contrast, in the present approach, all agents have the same information and there is no cost differential between forecasting strategies\(^6\).

The specification of a linear weighting in (1) is restrictive, so we consider convex $w(\cdot)$'s, following Hofbauer and Weibull (1996) Compared to the linear case, under convex weighting agents switch more aggressively to strategies that yield small squared errors. The convexity of the weighting function turns out to be a key factor determining whether persistent heterogeneity of forecasts is a possibility.

The analysis focuses on three forecasts that satisfy rationality\(^7\) within a standard asset pricing market. The fundamentalist forecast determined solely by expected future dividends, as dictated by the EMH, has special status in that there is always a small fraction of agents using this forecast. Agents may also choose a rational bubble or mystic forecast, which satisfies the weak efficient markets condition that returns are unforecastable. The reflective forecast follows the literature\(^8\).

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\(^6\) Branch and Evans (2006) study a cobweb model with two costless mis-specified forecasts.

\(^7\) The heterogeneous expectations models surveyed in Hommes (2006) typically postulate a number of bounded rational forecasting strategies such as trend-chasing. Combining such strategies with the imitative dynamics described here is potentially interesting, but, for the present work, we restrict our focus to strategies satisfying rational expectations.

\(^8\) Bates and Granger (1969) and Granger and Ranathan (1984) discuss the potential benefits. Elliot and Timerman (2008) has references such as Stock and Watson (1999) that empirically verify that combining forecasts can
that sets out an econometric view of the merits of combining forecasts and uses a weighted average of the other two forecasts. Furthermore, in an environment with heterogeneous expectations, the reflective forecast is the one that embodies the information about the alternative forecasts and the fractions of the agents using them.

The central question of the paper is whether the fundamental forecast is robust to the introduction of agents experimenting with alternative forecasts such as mysticism when the reflective forecast is an alternative forecasting strategy. If all forecasting strategies besides fundamentalism and reflectivism are driven from the model, these two forecasts coincide, and all agents use the fundamental forecast. Given sufficient bounds on the stochastic innovations in the model and given sluggish adjustment of the agents’ choices of forecasting strategies, the fraction of followers of reflectivism is increasing over time. Hence, the system is stable at the point where the share using reflectivism is at its maximum and all agents are using a forecast corresponding to the EMH.

However, even in this framework favorable to reflectivism, the elimination of alternative strategies such as mysticism is not a foregone conclusion. Given sufficient convexity of the weighting function \( w(\cdot) \) relative to the magnitudes of the payoffs in (1), when agents are more aggressively pursuing better performing strategies, reflectivism is not robust to the introduction of followers of mysticism. The potential for persistent heterogeneity, which can be viewed as an explanation of endogenously arising bubbles, is established by both analytic and simulation results.

\section{Asset Pricing}

A simple asset pricing model motivates a number of the important concepts. For an asset price \( y_t \), the notional model is a basic recursion

\[ y_t = \alpha y_{t+1}^F + u_t, \]

where \( \alpha < 1 \) is a discount factor and \( u_t \) is an income flow.\(^9\) This model has a solution under rational expectations based only on fundamentals (current and expected future dividends here), but (2) also admits rational bubble solutions that depend on extraneous variables. Convergence improve performance. The inclusion of the reflective forecast distinguishes the present paper from Guse (2008) who uses an imitative dynamic to describe switching between a fundamental forecast and one based on a sunspot.\(^9\)

\(^9\)This equation could equally apply to aggregate prices, exchange rates, etc..
to agreement on a single expectation \( y^*_t \) is viewed here as a possible conclusion rather than as an assumption. Given heterogeneous expectations and mean-variance optimizing agents, Brock and Hommes (1998) discuss conditions\(^{10}\) under which the realized security price depends on the weighted average of agents’ expectations.\(^{11}\) They show that

\[
y_t = \alpha \cdot x_t \cdot e_t + u_t,
\]

where \( e_t \) is a vector of the forecasts of \( y_{t+1} \) and \( x_t \) is a vector of the fractions of the population using each forecast. The discounted present value of the expected income stream

\[
y^*_t = u_t + \sum_{j=1}^{\infty} \alpha^j E_t(u_{t+j} \mid t),
\]

where \( \mathcal{I} \) is an information set available to the agents, will serve as a point of reference as it satisfies the strong version of the EMH, though it is not the unique solution to (2). The natural candidate for the fundamental forecast of \( y_{t+1} \) is thereby

\[
e_{2,t} = E(y^*_{t+1} \mid t) = \sum_{j=1}^{\infty} \alpha^{j-1} E(u_{t+j} \mid t).
\]

While it is common to assume that all agents somehow recognize \( e_{2,t} \) as the appropriate forecast, we make the following less restrictive assumption.

**Assumption 1:** The fraction \( x_{2,t} \) of the agents using the fundamental forecast \( e_{2,t} \) is bounded from below by \( \delta_2 > 0 \) for every \( t \).

These unyielding fundamentalists might well be impressed by the fact that (5) is touted by a large fraction of the academic literature in economics and finance. They do not need to take a position on, for example, the merits of transversality conditions vs. minimum state variables as a basis for (5) to recognize that (5) is prominently featured in Cochrane (2001) and McCallum (1983, 1997).

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\(^{10}\) In particular, there is a constant supply of a risky asset, and agents have a common belief about the variance of the returns.

\(^{11}\) There are several important differences between our analysis and Brock and Hommes (1998). Our choice of strategies differs from theirs. Brock and Hommes take payoffs to equal trading profits, which are a linear rather than concave function of forecast errors. Their discrete choice updating mechanism, as we note in the introduction, does not allow for convergence to a single expectation.
Our theoretical results allow for an arbitrary number of alternative forecasting strategies, but for the model of asset pricing we select a challenger to the fundamentalist forecast from among the martingale or rational bubble solutions to the model (2). This alternative forecast will be

\[ e_{3,t} = e_{2,t} + \alpha^{-t-1} m_t = E(y_{t+1}^* | t) + \alpha^{-t-1} m_t. \]  

(6)

where \( m_t = m_{t-1} + \eta_t \) is a martingale. We label this forecast mysticism because, while \( \alpha^{-t} m_t \) is thought by economic theorists to be extraneous, agents believing in (2) cannot rule out a martingale solution on the basis of that mathematical model.\footnote{Even an auxiliary belief in a stationary solution does not rule out a martingale in a finite number of periods. It is not possible to know with certainty that \( m_t \) is nonstationary from a finite data sample. In fact, when mysticism grows in popularity, that growth will often occur in the first few periods, well before tests for nonstationarity have useful power.} Followers of the mystical forecast might, for example, sincerely believe that \( \alpha^{-t-1} m_t \) is a valid addition to the fundamentals for \( y_t \). In fact, the martingale does influence \( y_t \) if the mystical forecast attracts followers.

Both the fundamental forecast (5) and mysticism (6) satisfy rational expectations in the homogeneous case, but this observation may not hold if there is heterogeneity in the choice of forecasting strategies in the population. Hence, the reflective forecast is postulated to be an average of the fundamental and mystic forecasts weighted according to their relative popularities.

\[ e_{1,t} = (1 - n_t) e_{2,t} + n_t e_{3,t} \]

where

\[ n_t = \frac{x_{3,t}}{x_{2,t} + x_{3,t}} \]

is the proportion of followers of the mystical forecast among those not following the reflective forecast. Such a forecast follows the literature on the benefits of combining forecasts. See Elliot and Timmerman (2008) for multiple references. The above ratio shows the need for the imposition of a minimum fraction \( \delta_2 \) using fundamentalism to avoid dividing by zero. Since reflectivism is based on other forecasts, there must be at least one other forecast in the population for reflectivism to be well specified.
The reflective forecast can be written as

\[ e_{1,t} = E(y_{t+1} \mid \tau) + \alpha^{-t-1} n_t m_t, \]  

(7)

which can be verified with the four previous equations. The martingale affects the reflective forecast according to the relative popularity of mysticism and fundamentalism. The realization for \( y_t \) can be obtained by substituting the expectations (5), (6), and (7) into (3), yielding

\[ y_t = y_t^* + \alpha^{-t} n_t m_t. \]  

(8)

The realization is thus the fundamentalist forecast plus, to the extent that some of the agents are following mysticism, a martingale term. Unlike the fundamental and mystic forecasts, both the realization (8) and the reflective forecast embody available information about the fractions of the population using the different strategies. Furthermore, in contrast to fundamentalism and mysticism, which satisfy rationality only in the homogeneous case, the reflective forecast satisfies rational expectations for any \( x_t = (x_{1,t}, x_{2,t}, x_{3,t}) \) describing the choice of forecasting strategies in the population.

Payoffs are given by the negative of the squared forecast error

\[ \pi_{i,t} = -(y_t - e_{i,t-1})^2. \]  

(9)

Evaluating forecasts using squared errors has a long tradition in econometrics\(^{13}\). In an asset pricing context, there are a number of choices for payoffs such as realized profits, excess returns or Sharpe ratios. For the present asset pricing model, Hommes (2001) shows that (9) is the natural objective function for mean-variance maximizing agents. If agents adjust for risk based on the variance of the profits, then forecast errors are a more appropriate payoff than realized profits.

The reflective forecast error

\[ U_t = y_t - e_{1,t-1} \]

\(^{13}\)Elliott and Timmerman (2008) discuss the role of mean squared prediction error and forecast combination in the literature on forecasting.
includes two components:

\[ U_t = (y_t^* - E(y_t^* \mid t-1)) + \alpha^{-t}(n_t m_t - n_{t-1} m_{t-1}). \]  

(10)

The first term on the right is the innovation in fundamentals. The second term on the right is the weighted martingale innovation. Using this approach, the payoff to reflectivism is

\[ \pi_{1,t} = -U_t^2. \]  

(11)

The payoffs to mysticism and fundamentalism also depend on \( A_{t-1} = \alpha^{-t}m_{t-1} \). Intuitively, \( U_t \) depends primarily on innovations and \( A_{t-1} \) depends on the level of the martingale and consequently is non-stationary. The fundamentalist forecast error

\[ y_t - e_{2,t-1} = U_t + n_{t-1} A_{t-1} \]

from (8) and (5) includes a fraction of the martingale term because, to the extent that some of the agents are following the mystical forecast, the realization (8) is affected by the martingale term. Note that if mysticism is driven out of the population so that \( n_{t-1} = 0 \), then the fundamental forecast coincides with the reflective forecast. The fundamentalist payoff is

\[ \pi_{2,t} = -U_t^2 - 2n_{t-1} U_t A_{t-1} - n_{t-1}^2 A_{t-1}^2. \]  

(12)

The mystical forecast error is

\[ y_t - e_{3,t-1} = U_t - (1 - n_{t-1}) A_{t-1} \]

from (8) and (6), and the resulting payoff is

\[ \pi_{3,t} = -U_t^2 + 2(1 - n_{t-1}) U_t A_{t-1} - (1 - n_{t-1})^2 A_{t-1}^2. \]  

(13)

Any of the three forecasting strategies could have the best payoff depending on the realizations of \( U_t \) and \( A_{t-1} \), as detailed in Table 1. If \( A_{t-1} \) is large relative to \( U_t \), then the third terms,
referred to as martingale terms, in the payoffs to fundamentalism (12) and mysticism (13) are the dominating feature causing both payoffs to under-perform reflectivism. The reflective forecast is constructed such that the martingale does not affect its payoff, so, when the martingale term is large, reflectivism is best. However, if \( A_{t-1} \) is not large and the "covariance" \( U_t A_{t-1} \) is large and positive, then mysticism could have the best payoff. Such an outcome corresponds to a fortunate (for the mystic) correlation between the martingale and the innovations in the model. Similarly, a large and negative covariance favors fundamentalism.

Despite these observations, reflectivism does have an inherent advantage over the other strategies, as one might expect given that the reflective forecast embodies extra information about the other forecasts and their fractions of supporters. Many evolutionary game theory dynamics depend on the fitness of the strategies, the difference between payoffs and the population average payoff \( \bar{\pi}_t \) given by \( \bar{\pi}_t = x_{1,t-1} \bar{\pi}_{1,t} + x_{2,t-1} \bar{\pi}_{2,t} + x_{3,t-1} \bar{\pi}_{3,t} \). In particular, the replicator dynamic with a linear weighting function \( w(\cdot) \) in (1) has the property that the fraction of agents using a strategy adjusts proportionally with the fitness of that strategy. Here, the population average is

\[
\bar{\pi}_t = -U_t^2 - \frac{x_{2,t-1}x_{3,t-1}}{x_{2,t-1} + x_{3,t-1}} A_{t-1}^2.
\]

While \( A_{t-1} \) does not enter the payoff \( \pi_{1,t} = -U_t^2 \) to reflectivism, if there is any heterogeneity in the population, it does affect the population average. Under the linear weighting according to population shares for \( \bar{\pi}_t \), the covariance terms in the mystic and fundamental payoffs cancel, so the population average payoff cannot be superior to the reflective payoff.

**Remark 1** Let \( \bar{x} = (1 - \delta_2, \delta_2, 0) \) denote the point where the fraction of agents following fundamentalism equals its minimum \( \delta_2 \) under Assumption 1 and the fraction of agents is using reflectivism equals its maximum \( 1 - \delta_2 \). For linear payoff weighting, which is \( w(\pi_{i,t}) = \pi_{i,t} \) in (1), the fitness of reflectivism \( \pi_{1,t} - \bar{\pi}_t \) is always positive, \( x_{1,t} \) is monotone increasing over time, and the point \( \bar{x} \) is stable.

Linear weighting is, of course, not the only possibility. Hofbauer and Weibull (1996, p. 563) note that curvature in \( w(\cdot) \) can change agent behavior in an evolutionary game somewhat analogously to how curvature in utility functions can affect behavior toward risk in other settings. The degree
of nonlinearity parameterizes a property that we interpret as agent aggressiveness. We interpret a
weighting function that places a relatively large weight on only small squared errors as symptomatic
of agent aggressiveness in pursuing accurate forecasting strategies. A weighting function that makes
a more moderate distinction between large and small squared errors characterizes less aggressive
agents.

A second consideration, enforcing the nonnegativity condition \( w(\pi_{i,t}) \geq 0 \), also motivates using
nonlinear weighting functions. Nonnegativity, which is common in static games, is desirable because
\( \bar{w}_t \) appears in the denominator of (1) and \( \bar{w}_t \leq 0 \) would be a problem. Weighting functions that
achieve nonnegativity are inherently nonlinear because no linear transformation of the squared error
payoffs will be unambiguously nonnegative.

We have two primary goals for the remainder of this paper. One is to identify conditions under
which we can turn Remark 1 into a rigorous proposition for nonlinear weighting functions. The
other is to explore via simulations conditions under which the point \( \bar{x} \) is not stable. It will turn out
that the curvature of the payoff weighting function and, hence, agent aggressiveness is important
in determining whether or not \( \bar{x} \) is stable.

3 Curvature and Selection Dynamics

We consider two particular nonlinear payoff weighting functions that enforce \( w(\pi_{i,t}) \geq 0 \) and
parameterize agent aggressiveness for the general dynamic (1).

**Truncation Weighting.** A simple way to achieve nonnegativity is to work with the weighting
\( w(\pi_{i,t}) = C + \pi_{i,t} \), where \( C \) is a constant chosen so that \( C + \pi_{i,t} > 0 \) for all strategies and all periods.
The revised replicator dynamic then becomes

\[
x_{i,t+1} - x_{i,t} = x_{i,t} \frac{\pi_{i,t} - \bar{\pi}_t}{C + \bar{\pi}_t}.
\]

In static game theory, the form of the replicator (15) is quite sufficient as it is easy to choose \( C \) to
be larger than the biggest payoff.\(^{14}\) Here, there is no lower bound for \(-U_i^2\) in the payoffs, but the

\(^{14}\) Hofbauer and Sigmund (1988, p. 133), Hofbauer and Sigmund (1998, pp. 76-7), Samuelson (1997, p. 66), and
Weibull (1997, pp. 122-3) discuss this version of the replicator dynamic.
truncation function
\[
w(\pi) = \begin{cases} 
C + \pi & \text{if } C + \pi \geq 0 \\
0 & \text{if } C + \pi < 0
\end{cases}
\] (16)
guarantees nonnegativity without requiring \(C + \pi_{i,t} \geq 0\) for all strategies and all periods.

The parameter \(C\) can be viewed as parameterizing agent aggressiveness. If \(C\) is small, then \(w(\pi) = 0\) for all but the smallest forecast errors because agents regard strategies with larger forecast errors as worthless. Smaller values for \(C\) increase the ratio \((\pi_{i,t} - \bar{\pi}_t)/(C + \bar{\pi}_t)\), causing a bigger change \(x_{i,t+1} - x_{i,t}\) for a given \(\pi_{i,t} - \bar{\pi}_t\) and \(\bar{\pi}_t\).

**Exponential Weighting.** The exponential transformation
\[
w(\pi) = e^{\pi/(2\sigma^2)}
\] (17)
achieves nonnegativity without sacrificing smoothness. The resulting dynamic is an example of a convex monotonic dynamic as discussed in Hofbauer and Weibull (1996).

Applying the exponential transformation to the squared forecast error \(\pi_{i,t} = -(y_t - e_{i,t-1})^2\) produces a familiar functional form
\[
w(\pi_{i,t}) = e^{-(y_t - e_{i,t-1})^2/(2\sigma^2)}.
\] (18)
This is a normal probability density. The “mean” of the forecast error \(y_t - e_{i,t-1}\) is zero and the “variance” is \(\sigma^2\). The parameter \(\sigma^2\) has no necessary relation to the statistical properties of the forecast errors. It instead determines how agents react to large and small squared forecast errors.

In the present context, \(\sigma^2\) parameterizes agent aggressiveness. If \(\sigma^2\) is large, \(w(\pi_{i,t})\) is not very sensitive to the magnitudes of the forecast errors. Equivalently, agents are not very aggressive about pursuing the best forecasting strategy. If \(\sigma^2\) is small, agents assign appreciable value to only the smallest forecast errors. Hofbauer and Weibull (1996, p. 563) describe the effect of convexity as “individuals react over-proportionally to higher payoffs,” as opposed to the replicator where the population shares adjust proportionally to fitness.
4 Dynamics

We are interested in the stability properties of a system that we can summarize as follows. In period $t$, the population fractions $x_{i,t}$, $i = 1, ..., k$, determine the choices among forecasts $e_{i,t}$, $i = 1, ..., k$, and the realization of the asset price

$$y_t = \alpha x_t \cdot e_t + u_t.$$ 

This is (8) above. Agents choose among the reflective forecast (7), the fundamentalist forecast (5), and the mystical forecast (6). The payoffs are determined by squared errors of the forecasts

$$\pi_{i,t} = -(y_t - e_{i,t-1})^2.$$ 

This is (9) above. It takes the specific forms (11), (12) and (13) for reflectivism, mysticism, and fundamentalism, respectively. The selection dynamic

$$x_{i,t+1} - x_{i,t} = x_{i,t-1} \frac{w(\pi_{i,t}) - \bar{w}_t}{\bar{w}_t}$$

produces the population fractions for period $t + 1$. This is (1) above. We consider two choices for $w(\pi_{i,t})$, truncation weighting (16) and exponential weighting (17).

The stability of the system depends on the parameters $C$ and $\sigma^2$ that characterize agent aggressiveness for the two cases. In Section 5 we explore some unstable regions using simulations. In this section, we establish a rigorous extension of Remark 1 for truncation weighting (16). The proof of stability shows that, given a bound on $U_t^2$ and a minimum fraction of agents following fundamentalism, for every $t$ either the fraction following reflectivism increases or mysticism is eliminated from the population. In either case, all agents end up following either the fundamental forecast or the reflective forecast, which are then identical.

The intuition for the proof can be described as follows. If the payoffs to all three strategies are greater than $-C$, then there is no truncation and the logic of Remark 1 is straightforward. For mysticism to attract followers, the payoff to fundamentalism must be below $-C$ and the payoff to mysticism must be best. However, for fundamentalism to perform so badly, $A_{t-1}^2$ must be large,
but if it is too large, the reflective payoff is greater than the mystic payoff. Therefore, for $U_t$ sufficiently bounded, the effect of the covariance term on the mystic payoff is limited, and there is no $A_{t-1}^2$ that is simultaneously large enough to force the fundamentalist payoff below $-C$ and small enough so that mysticism outperforms reflectivism.

**Proposition 2** There exists a constant $\varphi \in (0,1)$ such that for the dynamics given by (16) and (1) and for the payoffs (11), (12) and (13), if $U_t$ satisfies the condition

$$U_t^2 < \varphi C,$$

then either $x_{1,t+1} > x_{1,t}$ or $x_{3,t+1} = 0$.

**Proof.** The bound on $U_t^2$ guarantees that $w(\pi_{1,t}) > 0$. If $w(\pi_{2,t}) > 0$ and $w(\pi_{3,t}) > 0$, then (14) implies $x_{1,t+1} > x_{1,t}$. The same inequality holds if both $w(\pi_{2,t}) = 0$ and $w(\pi_{3,t}) = 0$. If $w(\pi_{2,t}) > 0$ and $w(\pi_{3,t}) = 0$, then $x_{3,t+1} = 0$.

Suppose then that $w(\pi_{2,t}) = 0$ and $w(\pi_{3,t}) > 0$. The case where $x_{2,t+1}$ hits its minimum $\delta_2$ is treated as a separate case. If $x_{2,t+1}$ is fixed, the $x_{1,t+1} > x_{1,t}$ if and only if $w(\pi_{1,t}) > w(\pi_{3,t})$ for any reasonable dynamic. This condition is equivalent to

$$2 (1 - n_{t-1}) U_t A_{t-1} - (1 - n_t)^2 A_{t-1}^2 < 0.$$  \hspace{1cm} (19)

Since $w(\pi_{2,t}) = 0$ in this case, it must be true that

$$C - U_t^2 - 2n_{t-1} U_t A_{t-1} - n_{t-1}^2 A_{t-1}^2 < 0,$$

but the condition from the proposition $U_t^2 < \varphi C$ implies that $C - U_t^2 > (1 - \varphi) C$. Combining inequalities yields

$$2n_{t-1} U_t A_{t-1} + n_{t-1}^2 A_{t-1}^2 > (1 - \varphi) C.$$

For this inequality to be true, there must be a minimum $A_{\min} > 0$ such that $|A_{t-1}| > A_{\min}$.

Now, we can show the inequality (19) guaranteeing $w(\pi_{1,t}) > w(\pi_{3,t})$ is satisfied in this case for a sufficiently small $\varphi$. If $U_t$ and $A_{t-1}$ have different signs, (19) is satisfied automatically. If they
have the same signs, (19) is satisfied as long as

$$|U_t| < \frac{1}{2} (1 - n_t) |A_{t-1}|$$

(20)

In this case, we have $|A_{t-1}| > A_{\text{min}} > 0$. Furthermore, $(1 - n_t)$ is strictly positive since $x_{2,t+1} \geq \delta_2$. Therefore, there exists a $\varphi \in (0, 1)$ such that for $U_t^2 < \varphi C$, the inequality (20) is satisfied and therefore so is $x_{1,t+1} > x_{1,t}$

For the case where $w(\pi_{2,t}) = 0$ and $w(\pi_{3,t}) > 0$ when $x_{2,t+1}$ is greater than its minimum, the condition for $x_{1,t+1} > x_{1,t}$ is $w(\pi_{1,t}) > n_{t-1} w(\pi_{3,t})$, which is weaker than the condition $w(\pi_{1,t}) > w(\pi_{3,t})$ for the case when $x_{2,t+1} = \delta_2$, so the argument above applies.

This proposition implies that the point $\bar{x}$ (Remark 1) satisfies a weak version of stability in the sense that if $x_t$ is in a neighborhood of $\bar{x}$, then it will remain there for future periods for an appropriate bound on $U_t$. The logic of the above proof is relevant for any point in the interior of the simplex for an appropriately chosen bound.

Proposition 2 shows that forecasts not corresponding to the EMH are eliminated if a sufficiently tight bound on the stochastic innovations can be established, relative to the aggressiveness of the agents represented by $C$. However, persistent heterogeneity of forecasts is also a viable possibility.

It is notable that a result such as Proposition 2 can be established in an environment allowing for heterogeneous expectations, but it is questionable that bounding stochastic elements such as dividend innovations is realistic. ‘Black swans’ in asset markets would violate such bounds, for example, as would normally distributed dividends, though such events could be rare. Simulation results in the next section help to clarify the quantitative importance of the formal analysis.

For agents using the exponential weighting function (18), we present an informal argument that a restriction on $U_t$ in relation to agent aggressiveness again guarantees the fraction of agents following reflectivism is increasing over time. For reflectivism to increase under the dynamic (1), the weighted payoff to reflectivism $w(\pi_{1,t})$ must be larger than the weighted average $\bar{w}_t$ payoff so

$$\frac{w_t}{w(\pi_{1,t})} < 1.$$  Using exponential weighting (17), this fraction may be written

$$\frac{\bar{w}_t}{w(\pi_{1,t})} = x_{1,t-1} + x_{2,t-1} \exp(\pi_{2,t} - \pi_{1,t}) + x_{3,t} \exp(\pi_{3,t} - \pi_{1,t}).$$
To examine the dynamics around the introduction of the mystic, assume the martingale is small and the payoff differences in the above equation are close to zero. The following uses Taylor approximations of the exponential functions around $A_{t-1} = 0$, eliminating powers of $A_{t-1}$ higher than two since they are small.

$$
\frac{w_t}{w(\pi_{1,t})} \approx 1 - (2\sigma^4)^{-1} \left( \frac{x_{2,t-1} x_{3,t-1}}{1 - x_{1,t-1}} \right) A_{t-1}^2 \left[ \sigma^2 - U_t^2 \right]
$$

(21)

If we restrict the Taylor approximation to linear terms, then $U_t^2$ does not appear, the ratio $\frac{w_t}{w(\pi_{1,t})} < 1$, and reflectivism’s share increases over time as with linear weighting. When the second-order term of the Taylor approximation is included, the sign of $\frac{w_t}{w(\pi_{1,t})}$ depends on the sign of $\sigma^2 - U_t^2$. Hence, if $U_t$ is appropriately bounded then $\sigma^2 - U_t^2$ is positive, but for $U_t^2 > \sigma^2$, $x_{1,t}$ decreases, opening the door to persistent heterogeneity in forecasting strategies. So, if the stochastic innovations in the model are sufficiently large relative to agents’ aggressiveness in switching to better performing strategies, reflective monotonicity does not hold and mysticism has an opportunity to gain adherents. Again, stability and persistent heterogeneity are both possible, depending whether there is a sufficient bound on the dividends relative to the aggressiveness of the agents. To determine the quantitative effects of changes in model parameters on the likelihood for such instability, we examine simulations of the asset pricing model.

5 Robustness

The results in the previous sections show the potential for either agreement on the fundamental forecast within the population or persistent heterogeneity in forecasting strategies. This section describes simulation results that quantitatively characterize the conditions that might lead to heterogeneous expectations by considering how the following for mysticism might increase from the initial minimal following\(^{15}\) $n_{t-1} \approx 0$. We then discuss the interpretation of the model, using it to describe bubbles and other features of financial markets data.

\(^{15}\)Binmore, Gale, and Samuelson (1995) introduce drift, which is a similar approach that examines the effects of introducing a small fraction using a strategy.
5.1 Simulations

The simulation results confirm the intuition from the formal discussion. For sufficiently bounded dividends and shocks to the martingale and sluggish switching between forecasting strategies, mysticism cannot attract a following. With mysticism playing little or no role in the dynamics, the reflective and fundamental forecasts coincide, and the model reduces to the homogeneous environment corresponding to the EMF. However, for sufficiently large shocks, mysticism can attract a significant following, and the extraneous martingale can have an impact on the asset price, as seen in equation (8) for \( n_t > 0 \). In such a situation, the asset price still satisfies a weak version of the efficient markets hypothesis in that prices and returns are not forecastable, but the strong version is not satisfied since information besides expected future dividends impacts the asset price.

The magnitude of the stochastic elements of the model is crucial. For mysticism to succeed, the term based on the innovations \( U_t \), see eq. (10), must be large relative to the curvature of \( w(\pi) \), which we interpret as aggressiveness of the agents. Furthermore, the term \( A_t \), which is based on the martingale, must be in a range where the covariance term \( U_t A_{t-1} \) in the mystic payoff (13) outweighs the impact of the third term with \( A^2_{t-1} \), as indicated in the proof of proposition 2.

Some normalization is necessary, and we set the standard deviation of the fluctuation in fundamentals, \( y_t^* - E(y_t^*| x_{t-1}) \) in (10), to \( \sigma^* = 1 \). We set the discount factor to \( \alpha = 0.99 \). The other two parameters are either \( C \) or \( \sigma^2 \) and the standard deviation \( \sigma_\eta \) of the martingale innovation \( \eta_t = m_t - m_{t-1} \). The innovation in fundamentals and the innovation in the martingale are both taken to be normally distributed, which means that the conditions in Proposition 2 and equation (21) for stability are not met for all \( t \). Whether this is quantitatively important is a key question to be examined through simulations.

The initial population share for fundamentalism is set to the minimum \( x_{2,0} = 0.05 \), but mysticism starts at \( x_{3,0} = 0.0001 \), which is 500 times smaller. At the start then, \( n_t = 0.002 \) and agents are very nearly following the fundamentalist forecast. Therefore, if mysticism cannot attract a much greater following then the martingale term has little impact and the asset price governed by the EMH. If \( x_{3,t} \) falls below 0.0001 in a given period, we restart the martingale at \( m_t = 0 \) and reset the fraction of the agents following mysticism to \( x_{3,t} = 0.0001 \).\(^{16}\) If the fractions given by the

\(^{16}\)It would be possible to have martingales starting more often, perhaps every period. That would complicate the simulations but would probably work to favor more frequent episodes of heterogeneous expectations.
unconstrained dynamics break these bounds, we set those fractions to their minima and allocate the other fractions so that \( x_{1,t} + x_{2,t} + x_{3,t} = 1 \) and the unconstrained fractions are in proportion to their weighted payoffs. For example, to make Assumption 1 operational, if equation (1) sets \( x_{2,t+1} < \delta_2 \), then we let \( x_{2,t+1} = \delta_2 \) and use equation (1) with only the other two forecasting strategies to determine the division of the remaining fraction of agents \( 1 - \delta_2 \). While this process is a bit tedious to describe, it is not of practical importance because the outcome is determined by the behavior of \( x_t \) in the interior of the simplex \( \Delta \).

We define robustness in terms of the probability that mysticism attains a specific percentage following. The simulations start at \( x_{3,0} = 0.0001 \). We calculate the probability that \( x_{3,t} \geq 0.20 \) at any time within the first 100 periods. If we frequently observe \( x_{3,t} \geq 0.20 \) within the first 100 periods, we conclude that, for the given parameter values, the tendency to converge to a single forecast is not robust to one agent in 10,000 experimenting with mysticism.

Table 2 (the truncation weighting function) and Table 3 (the exponential weighting function) report results for these probabilities and show rather forcefully that mysticism will be an important factor if agents are sufficiently aggressive and the martingale innovations are sufficiently large. While convergence in Table 2 to a single expectation is common for \( C = 16 \) and likely for \( C = 8 \), the probability of episodes of mysticism approaches one for smaller values of \( C \). In Table 3, if the convexity parameter \( \sigma^2 \) is \( 1/4 \) or smaller and the martingale innovation standard deviation \( \sigma_\eta \) is at least \( 1/8 \), then the probability of significant episodes of mysticism ranges from over one-half to 1.000. Together, Tables 2 and 3 tell a consistent story. If agents are aggressive, which we can identify as \( \sigma^2 \leq 1/4 \) or \( C \leq 4 \), then the outcome is not well characterized as convergence to homogeneous expectations. The probability of repeated episodes of mysticism approaches 1.000 for some parameter combinations.

For martingale innovations with standard deviation above \( \sigma_\eta = 2 \), however, the frequency of mystic success declines. For large levels of \( A_{t-1} \), the reflective payoff (11) is superior, since the large martingale terms involving \( A_{t-1}^2 \) in the payoffs to fundamentalism (12) and mysticism (13) overwhelm the covariance terms involving \( U_t A_{t-1} \). This intuition is also the reason that only a restriction on \( U_t \), and not \( A_{t-1} \) or the payoffs, is necessary for stability in Proposition 2.

Tables 4-7 give a snapshot of the behavior of the model after 100 periods. Tables 4 and 6 show the mean fraction of mystic followers \( x_{3,100} \) for the linear and exponential weighting cases, while
Tables 5 and 7 show the standard deviations of those values over 10,000 trials for the two cases. In cases where mysticism could not gain followers, the mean values are at or close the starting value of 0.0001 with minimal variation. However, when mysticism has a chance, the means are well above the minimum and show great variation in the possible outcomes, implying that it is not hard to find instances where mysticism is the dominant strategy.

As noted in the introduction the timing of the general dynamic (1) is different than some standard developments of the replicator. For example, Samuelson (1998, p. 64) discusses a dynamic of the form

$$x_{i,t+1} = x_{i,t} \frac{w(\pi_{i,t})}{\overline{w}_t},$$

(22)

where $\overline{w}_t = x_{1,t}w(\pi_{1,t}) + x_{2,t}w(\pi_{2,t}) + \cdots + x_{n,t}w(\pi_{n,t})$, whereas the dynamic (1) lags the right hand side population shares $x_{i,t-1}$, a natural step given that the time $t$ payoffs $\pi_{i,t}$ depend on time $t-1$ strategy choices represented by $n_{t-1}$ in the payoffs to fundamentalism (12) and mysticism (13). Simulations of the model using the more standard dynamic (22) give almost identical outcomes to those presented here, so the results are not dependent on the choice of timing.

5.2 Interpretation

The model with outbreaks of mysticism represents an appealing model of a bubble. Although mysticism can gain a following for certain parameter values, it cannot last indefinitely. If mysticism is at its maximum, the presence of the minimum fraction following fundamentalism ensures that the mystic and reflective forecasts are not identical. Although the mystic forecast can outperform the reflective forecast for given periods, the expected payoff of the reflective forecast is superior to the expected payoff to mysticism so agents eventually abandon mysticism. Hence, bubbles arise and collapse endogenously in contrast to the models found in the literature on rational bubbles\textsuperscript{17}. Furthermore, the reflective forecast is an apt description of bubble psychology. Agents adopt a forecast based on information they may know to be extraneous because other agents are using that information.

It might be argued that the existence of a minimum fraction of fundamentalists is unlikely given that mysticism can have significant periods of success. However, if all agents adopt the mystic

\textsuperscript{17}See, for example Evans (1991). See Parke and Waters (2007) for further discussion and simulation examples.
forecast the model collapses to a never-ending rational bubble, which is economically implausible. An alternative interpretation is that the small fraction of unyielding fundamentalists is all that is necessary to ensure that the asset price remains connected to the dividends in the long run.

The conclusion that extraneous information embodied in the mystic forecast can affect asset prices depending on magnitude of the shocks and the aggressiveness of the agents has both theoretical and quantitative support. Hence, asset price bubbles arising from the presence of mysticism are a definite possibility, but so are stretches of time where the EMH is satisfied. The shocks to the fundamentals are closely related to the uncertainty about future dividends, suggesting that bubbles are more likely to arise for the stocks of firms in new industries or for recently developed asset classes, two areas when judgements about future profits and performance are difficult.

Parke and Waters (2007) show that, when conditions are favorable for repeated episodes of mysticism, the observed excess returns will exhibit volatility clustering. Standard tests for autoregressive conditional heteroskedasticity (ARCH) reject the null of homoskedasticity. The widespread finding of ARCH effects in empirical work could thus be taken as supporting the notion that aggressive agents are precluding convergence to a single expectation.

6 Summary and Conclusions

Our evolutionary game theory approach demonstrates that agreement on a unique rational expectation and the persistent presence of heterogeneous forecasts based on extraneous information are both distinct possibilities. The primary determining factor is the aggressiveness of the agents in switching forecasts relative to the magnitude of the stochastic elements. Convexity in the weighting function of the payoffs in the evolutionary game theory dynamic is directly related to the switching speed. Sluggish adjustment and small shocks correspond to stability of the point where mysticism is eliminated and agents coordinate on a forecast corresponding to the efficient markets hypothesis. Higher aggressiveness gives rise to the possibility that mysticism could play a significant role in asset price dynamics, potentially leading to persistent deviations from the fundamental forecast.

These results have a direct bearing on the merits of simply assuming that all agents agree on a single forecast. The theoretical results imply that such an assumption may be a reasonable abstraction for agents not overly aggressive in pursuing the best forecast, but also imply that an
environment populated by aggressive agents may be fertile ground for emergence of heterogeneous expectations. The simulation results confirm that, for our asset pricing example, sufficient agent aggressiveness can lead to persistent heterogeneous expectations.
<table>
<thead>
<tr>
<th>Realized $U_t$</th>
<th>Best Payoff</th>
<th>Middle Payoff</th>
<th>Worst Payoff</th>
</tr>
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<td>$U_t &lt; -n_{t-1}A_{t-1}/2$</td>
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<td>Refl.</td>
<td>Myst.</td>
</tr>
<tr>
<td>$-n_{t-1}A_{t-1}/2 &lt; U_t &lt; (1 - 2n_{t-1})A_{t-1}/2$</td>
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<td>Fund.</td>
<td>Myst.</td>
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<td>$\sigma_\eta$</td>
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<td>$C = 8$</td>
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<td>---------</td>
<td>---------</td>
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<tr>
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Table 3

<table>
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<tr>
<th>Exponential Weighting (17)</th>
<th>Probability $x_{3,t} \geq 0.20$ for some $t \leq 100$</th>
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<tbody>
<tr>
<td>10,000 trials</td>
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<tr>
<td>$\sigma_\eta = 1/16$</td>
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Table 4
Truncated Squared Errors (16)
Mean of $x_{3,100}$

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<th>$\sigma_\eta$</th>
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Table 5
Truncated Squared Errors (16)
Standard Deviation of $x_{3,100}$

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### Table 6

**Exponential Weighting (17)**

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### Table 7

**Exponential Weighting (17)**

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References


